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LETTER TO THE EDITOR

Flexural elasticity of percolation lattices

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Abstract. Using a two-dimensional beam lattice, we study the critical behaviour of the flexural elastic modulus around the percolation threshold, both experimentally and theoretically. We show that this case can be related to an anisotropic electrical conductivity percolation problem. This provides proof that the flexural elastic modulus in the vicinity of the percolation threshold scales with a critical exponent smaller than (or equal to) the usual conductivity exponent, which should be contrasted with in-plane deformation where the critical exponent is known to be much larger. The experimental data is consistent with this result.

The mechanical behaviour of randomly depleted media has recently been the subject of considerable interest since it has been recognised that the elastic transport properties were markedly different from other scalar transport properties (e.g. electrical or thermal conductivity) [1] (for a brief review see [2]). More precisely, in the framework of percolation, we know that the conductivity, G, of a lattice goes to zero when the proportion of bonds present tends to a threshold value p_c, as

$$G \propto (p - p_c)^t \qquad (1)$$

whereas the elastic modulus, E, approaches zero as

$$E \propto (p - p_c)^\tau . \qquad (2)$$

Specifically, for the two-dimensional case, t and τ have been estimated to be $t = 1.300 \pm 0.005$ [3] and $\tau = 3.96 \pm 0.04$ [4] thanks to complicated numerical simulations. Experimental data obtained so far (see e.g. [5]) are consistent with these values. However, one should note that for the two-dimensional case, the above-mentioned results concern the case of *in-plane* deformation.

We report here a study of *out-of-plane* deformation, namely the case of the flexion of a grid randomly degraded, around the percolation threshold. We will show that, in

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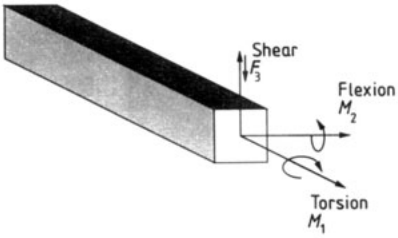

Figure 1. The contribution of forces and torques to the bending of a lattice out of the plane with the corresponding orientations of the axes used in the text.

contrast to in-plane deformation, the flexural elastic modulus, F, vanishes at threshold according to a power-law

$$F = (p - p_c)^{\tau_f} \tag{3}$$

where τ_f is a critical exponent which will be shown to be smaller than t, the conductivity critical index:

$$\tau_f \leqslant t. \tag{4}$$

We model our grid by a lattice of beams. Such a model has already been considered in the case of percolation [6, 7]. It allows one to introduce the bending contribution inisde a bond as a particular two-body interaction and to avoid introducing other kinds of bonds (like angular springs whose status is ill-defined in the degradation process). Moreover as part of the basic concepts of the strength of materials (a detailed discussion of the analogy of different bond-bending situations can be found in [7]), the elastic behaviour of a beam has been widely tested and used.

The most general form of the elastic energy, E, of a beam (one bond of the lattice) can be written as [8]:

$$E = \tfrac{1}{2}\int \frac{F_1^2}{ES} + \frac{F_2^2}{GS} + \frac{F_3^2}{GS} + \frac{M_1^2}{GI_1} + \frac{M_2^2}{EI_2} + \frac{M_3^2}{EI_3}\,\mathrm{d}x \tag{5}$$

where x is an abscissa along the axis of the beam, 1, 2 and 3 are three axis, 1 along the axis of the beam, 2 perpendicular to it and in the plane of the lattice, and 3 normal to the plane (figure 1). $F_i(x)$ is the ith component of the force, and $M_i(x)$ is the ith component of the moment. E and G are respectively the Young and shear moduli of the material. Finally S and I_i are geometrical factors (area and geometrical inertia of a transverse section). One should also note that, if no external force is applied to the beam apart from at its ends, then equilibrium requires that $F(x)$ is constant (independent of x) but $M(x)$ is affine with x:

$$F_2 = \mathrm{d}M_3/\mathrm{d}x \tag{6a}$$

$$F_3 = -\mathrm{d}M_2/\mathrm{d}x. \tag{6b}$$

We see that the elastic behaviour is naturally decoupled between in-plane deformations (which involve F_1, F_2, and M_3) and out-of-plane ones (which depend on F_3, M_1, and M_2) (see figure 1). We will now only deal with the second case and therefore we set $F_1 = F_2 = M_3 = 0$ from now on. Another important feature is that, for a loopless structure, it is possible to propagate only a flexural moment if this moment is in the plane of the lattice. In the lattice, the forces F will be identically zero and thus the moments will be constant along the bonds. The experiment discussed previously would be one example of such a case if it did not contain loops, since pure torques are applied onto opposite sides of the grid.

We can however use this property in order to obtain a bound on the critical exponent characterising the vanishing of the flexural modulus. If we impose the condition that the forces are identically zero inside the lattice, ($F_3 = 0$ along all bonds), then we can construct an admissible field of stress in the lattice that balances the applied torque, in such a way that only local torques exist. (Admissible means that local equilibrium is fulfilled at each node.) We can also use the symmetry of the square lattice for simplicity of the argument. If the applied torques at the edges of the grid are along one axis of the square lattice (say direction A) then the only stress that any bond inside the grid will carry will be a moment oriented along the same direction A, under the hypothesis that $F_3 = 0$. So, if the bond is along A, it will be submitted to a torsion, whereas if it is perpendicular to A, it will be subject to a flexion out of plane. Therefore a single scalar will now characterise the state of stress of each bond. The problem is considerably simpler than it seemed to be when considering the full energy expression.

We now want to minimise the energy on all admissible fields of stress that satisfy our restriction. We end with the following simple problem: each bond will carry a moment along direction A. We have to minimise the total energy—i.e. the sum of the square of the moments carried by each bond times the local elastic modulus, $1/GI_1$ if it is oriented along A or $1/EI_3$ if it is perpendicular—under the constraints of fulfilling the equilibrium of each node and respecting the boundary conditions. The equilibrium of a node reduces here to the zero sum of moments carried by neighbouring bonds. The boundary condition expresses the fact that the imposed torque is balanced by the sum of moments of bonds reaching the side concerned. This problem is thus formally identical to an anisotropic conductivity problem, where the ratio of the conductivities along the two axis of the lattice is GI_1/EI_3. We can thus use the following correspondence: current $\equiv$ moment; voltage $\equiv$ angle of rotation; dissipative energy $\equiv$ potential energy; conductivity $\equiv$ elastic modulus; equilibrium $\equiv$ conservation of current.

We therefore encounter a problem of the conductivity of an anisotropic lattice at the percolation threshold. It has been demonstrated [9] that the system loses its anisotropy around the percolation threshold and thus scales with the usual conductivity index t. Thus, we can bound the real elastic energy of the lattice by that computed by our admissible state of stress:

$$E \leqslant \tfrac{1}{2}M_{\mathrm{f}}^2/K \tag{7}$$

where M_{f} is the applied torque, and K is a flexural elastic modulus obtain through our correspondance. Thus K is also a conductivity which scales as

$$K \propto (p - p_{\mathrm{c}})^{t}. \tag{8}$$

We can conclude that the exponent governing the singularity of the flexural elastic modulus at threshold is

$$\tau_{\mathrm{f}} \leqslant t = 1.30 \pm 0.01. \tag{9}$$

Let us note that this result is not restricted to the case of the square lattice; however, we have used this case for simplicity of the argument. This inequality is consistent with the experimental data presented below.

A similar correspondence has already been used [10] for in-plane deformation to demonstrate that the combination $t + 2\nu$, where ν is the correlation length exponent, is an upper bound for τ, although it seems to be very close to the most precise numerical

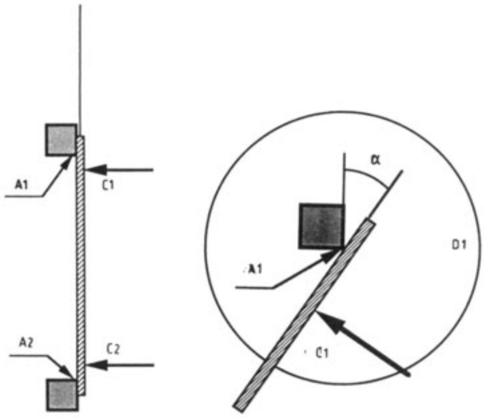

Figure 2. (*a*) Schematic representation of the experimental set-up. (*b*) Details of the application of forces on the grid. Key: A1, A2, axes of rotation; C1, C2, horizontal knives applying bending moments; D1, disc; α, angle of rotation of the disc.

estimate. From our result (equation (9)), we can conclude that the in-plane and the out-of-plane modes of deformations scale very differently. A hint of this has already been given in [11].

This somewhat surprising result might have very curious consequences. For example, for p close to 1 (i.e. very few missing bonds) the in-plane elastic modulus is much larger than for out-of-plane. However, as the critical indices for both problems are very different close enough to the threshold the in-plane stiffness should be the weaker. It implies that for most experimental conditions the crossing of these moduli should occur very close to p_c. The vibrational eigenfrequencies and eigenmodes of the system should suffer from this proximity: if one can expect to distinguish easily between in- and out-of-plane deformation far from the threshold, this will no longer be true at the crossing point where most modes will be a mixture of both kinds of deformation (see e.g. [11]). Buckling instabilities should also be strongly affected by this difference in scaling.

The lattice used in the experiments is a square gird, made of steel. The bonds are 1 mm thick and 2 mm wide. The lattice mesh is 9 mm. The area of the lattice is 66×66. A fraction $1 - p$ of bonds is cut at random. The values of p corresponding to the experimental measurements were 0.625, 0.6, 0.575, 0.55 and 0.525. For each of these values, both the conductivity G and the elastic modulus E are measured.

G is obtained using a four-probe method with a constant current of 1 A, and a DC voltmeter (resolution 1 μV). The resistance of the grid varies from 1.78 mΩ to 15.8 mΩ when p goes from 0.625 to 0.525. The resistance of the intact lattice is 0.38 mΩ.

The elastic modulus is measured in the following way. The grid is hanging from two wires linked to its upper end. In its vertical position, it is hanging freely close to two horizontal square bars, as shown in figure 2(*a*). The bending moments are applied to the grid by two horizontal knives (C1 and C2) whose lengths are equal to that of the grid. These knives are always perpendicular to the plane of the lattice since they are constrained to rotate around two axes, of centre A1 and A2 (see figure 2(*b*)), because they are attached rigidly to two discs D1 and D2 onto which a tangential force is imposed. In this way, well controlled bending moments are imposed on the grid. We measure the angle of rotation α of one disc, D1, from the variation of the electrical resistance of a potentiometer, the axis of which is linked to the axis A1.

Before giving the experimental results, it seems useful to make some comments on the difficulties encountered during the mechanical measurements: the process of cutting the bonds, using nippers or a milling machine, progressively produces some deformations

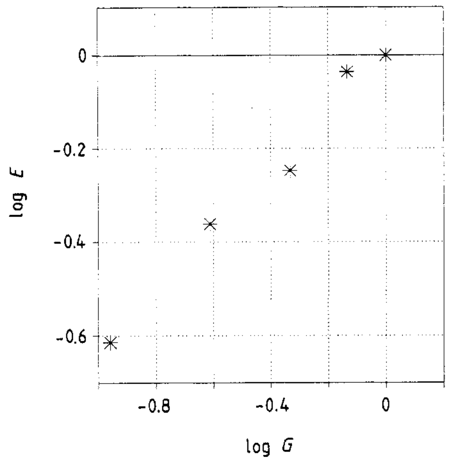

Figure 3. Log–log plot of the flexural elastic modulus against conductance of the lattice for different values of p. The data is expected to follow a power-law behaviour close to threshold, with an exponent equal to the ratio τ_f/t.

of the grid. These unavoidable deformations are both in plane and out of plane. The initially plane grid progressively acquired some bending. These deformations were such that a first series of data obtained on a sample could not be used. During the experiment, an unknown part of the applied torque was used to balance the overall bending of the grid, and to bring the sample back into contact with the knives C1 and C2, and the vertices A1 and A2 of the horizontal bar shown in figure 2(*a*). Therefore, the values of the elastic flexural modulus were erroneous.

In a second sample, we prevented the previous overall bending by screwing the part of the grid in contact with the knives onto a thick (10 mm) metal bar. Due to our experimental procedure, we needed to 'reshape' the lattice in such a way that the geometry of the remaining bonds was identical with that of the initial (undamaged) lattice. Thus, a more or less important work-hardening took place, resulting in some experimental errors in the determination of the elastic modulus. The importance of these errors however decreases as one gets closer to the percolation threshold. For that reason, we restricted ourselves to measurements in the range of p-values between 0.625 and 0.525 only.

When the applied torque is increased progressively from zero, the rotation angle first increases linearly, before curving down. For low torques, the slope of this curve is well defined. It is the inverse of the elastic bending modulus. This quantity, normalised to its value at $p = 0.625$, is plotted as a function of the electrical conductivity in figure 3.

From the data it is impossible to extract any estimate for τ_f. However, we note that the conductivity decreases faster than the flexural modulus as one gets closer to p_c, which is consistent with the bound (equation (9)) obtained above.

In conclusion, we have shown theoretically that the flexural elasticity of a percolation two-dimensional lattice has a critical exponent smaller or equal to that of conductivity, contrary to the in-plane deformation elasticity. This prediction is supported by experimental measurements.

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